ELEN 4810 Midterm Exam

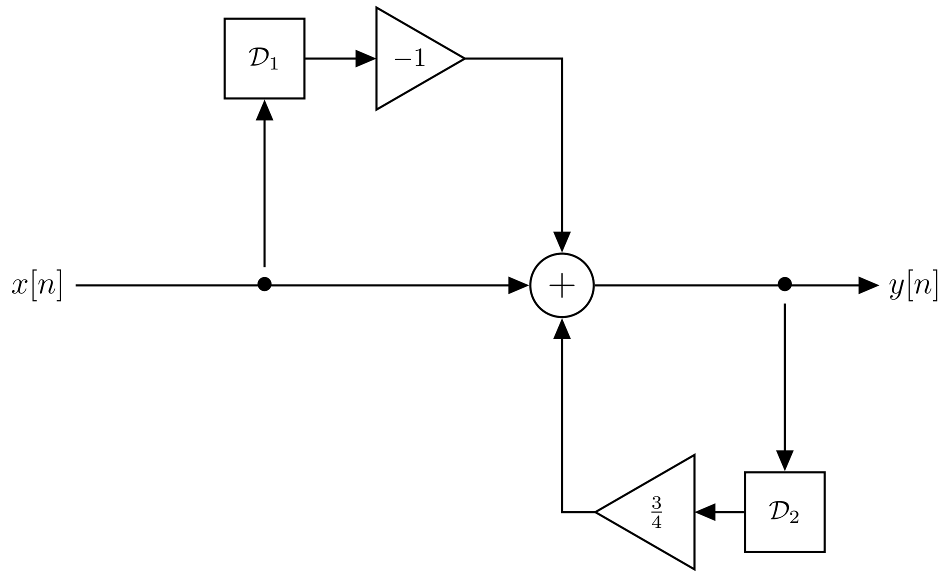
Monday, October 31, 2022, 1:10-2:40 PM. One sheet of handwritten notes is allowed. No electronics of any kind are allowed. Please record your answers in the exam booklet. Raise your hand if you need additional scratch paper.

There are a total of 3 questions. Good luck!

Name: SOLUTIONS

Uni:

1. Systems in Time and Frequency. Consider the causal linear, time invariant system corresponding to the following block diagram:



Here, $\mathcal{D}_1$ denotes an ideal delay by one sample, $\mathcal{D}_2$ denotes an ideal delay by two samples, and the triangular blocks denote multiplication by scalars -1 and $\frac{3}{4}$, respectively.

(a) Is the system stable? Why or why not?

(b) What is the frequency response $H(e^{j\omega})$ of the system?

(c) What is the output of the system when the input is $x[n] = (-1)^n$? What about when $x[n] = 1$ for all n?

(d) Suppose that the output of the system is $y[n] = 16(-1)^n$. What are the possible values of $x[n]$? Please make your answer as broad as possible for full credit.

Answer to Problem 1:

(a). Yes, the system is stable. The system consists of a series concatenation of two blocks: an FIR system, and an IIR system with impulse response $h[n] = (3/4)^n u[n]$. This is absolute summable, and so both subsystems are stable.

(b). Notice that $y[n] = 3/4y[n-2] + x[n] - x[n-1]$. Hence,

$$\left(1 - \tfrac{3}{4}e^{-j2\omega}\right)Y(e^{j\omega}) = \left(1 - e^{-j\omega}\right)X(e^{j\omega}).$$

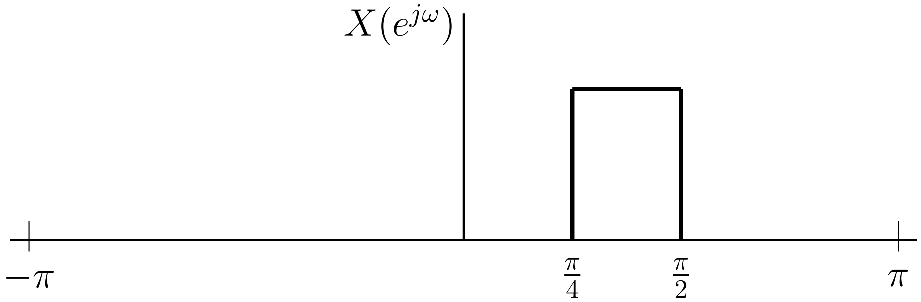
We have

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1 - e^{-j\omega}}{1 - \frac{3}{4}e^{-j2\omega}}. \tag{1}$$

(c). When $x[n] = (-1)^n$, $y[n] = H(e^{j\pi})(-1)^n = 8(-1)^n$. Similarly, when $x[n] = 1$ for all n, $y[n] = H(e^{j0}) \times 1$ for all n; notice that $H(e^{j0}) = 0$, and so $y[n] = 0$.

(d). Any $x[n]$ of the form $x[n] = 2(-1)^n + c$ for $c \in \mathbb{C}$ produces this output $y[n]$.

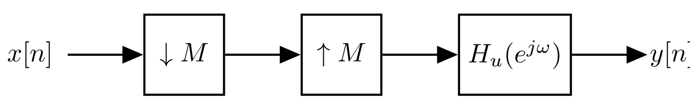
2. Sampling and Downsampling. A bandlimited continuous-time signal $x_c(t)$, with bandlimit Ω_M is sampled with rate Ω_s to produce a discrete-time signal $x[n]$. The below graph shows the DTFT $X(e^{j\omega})$.



Please answer the following questions:

Part (a). Using the graph of $X(e^{j\omega})$, **please determine the ratio** Ω_s/Ω_M of the sampling rate Ω_s to the bandlimit Ω_M.

Part (b). Consider the following block diagram:

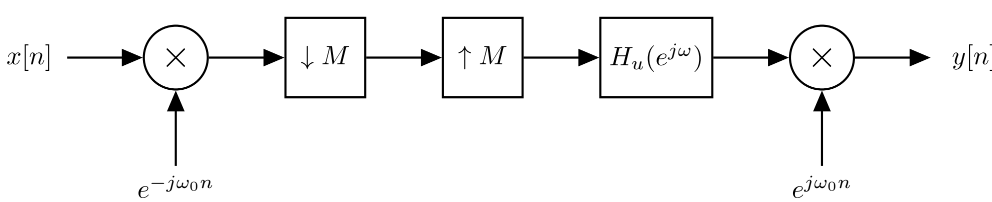


Here, $H_u(e^{j\omega})$ is the ideal upsampling filter

$$H_u(e^{j\omega}) = \begin{cases} M & |\omega| < \frac{\pi}{M} \\ 0 & \frac{\pi}{M} \leq |\omega| \leq \pi \end{cases}$$

What is the largest integer M for which $y[n] = x[n]$?

Part (c). Now consider the following modified block diagram:



Can you increase M by choosing ω_0 appropriately? **Please specify the largest possible M and corresponding best choice of ω_0.**

Answer to Problem 2:

(a). $\Omega_s = 4\Omega_M$ – the interval $[-\Omega_s/2, \Omega_s/2]$ maps into $[-\pi, \pi]$. The largest frequency here is $\omega = \pi/2$.

(b). $M = 2$, since X is bandlimited to $[-\pi/2, \pi/2]$.

(c). Setting $\omega_0 = 3\pi/8$, we obtain a modulated signal $x[n]e^{-j\omega_0 n}$ which bandlimited with bandlimit $\pi/8$. For this signal, we can set $M = 8$.

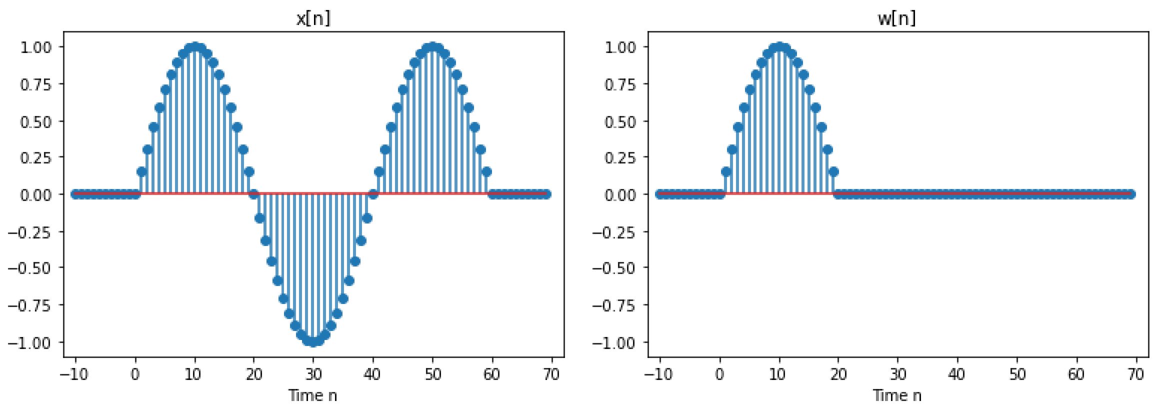
3. Correlation and Convolution. Consider discrete-time signals

$$x[n] = \begin{cases} \sin\left(\frac{\pi n}{20}\right) & 0 \le n \le 60 \\ 0 & \text{else} \end{cases} \tag{2}$$

and

$$w[n] = \begin{cases} \sin\left(\frac{\pi n}{20}\right) & 0 \le n \le 20 \\ 0 & \text{else} \end{cases} \tag{3}$$

The signals x and w are plotted below:



Let $\tilde{w}[n] = w[-n]$, and let y be the convolution of $\tilde{w}$ and x: $y = \tilde{w} * x$.

Part a: Where are the nonzeros? The nonzero entries of $y[n]$ satisfy $N_1 \le n \le N_2$. Please determine N_1 and N_2 – make N_1 as large as possible and N_2 as small as possible.

Part b: Where are the maximum / minimum values? For which value/values of n is $y[n]$ maximized? For which value/values of n is $y[n]$ minimized?

Answer to Problem 3:

(a). The nonzeros of $x[n]$ satisfy $a \le n \le b$, with $a = 1$ and $b = 59$. The nonzeros of $\tilde{w}[n]$ satisfy $c \le n \le d$, with $c = -19$ and $d = -1$. Hence, we can take $N_1 = a + c = -18$ and $N_2 = b + d = 58$.

(b). We have $y[n] = \langle s_n w, x \rangle$. This is maximized at $n = 0$ and $n = 40$, and minimized at $n = 20$.

Scratch paper: